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The hamiltonian index of a graph and its branch-bonds☆

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Abstract

Let G be an undirected and loopless finite graph that is not a path. The smallest integer m such that the iterated line graph $L^m(G)$ is hamiltonian is called the hamiltonian index of G, denoted by $h(G)$. A reduction method to determine the hamiltonian index of a graph G with $h(G) \geqslant 2$ is given here. We use it to establish a sharp lower bound and a sharp upper bound on $h(G)$, respectively, thereby improving some known results of Catlin et al. [J. Graph Theory 14 (1990) 347] and Hong-Jian Lai [Discrete Math. 69 (1988) 43]. Examples show that $h(G)$ may reach all integers between the lower bound and the upper bound. We also propose some questions on the topic.

MSC: 05C45; 05C38

Keywords: Hamiltonian index; Branch-bond; Reduction method; Iterated line graph

1. Introduction

We use [2] for terminology and notation not defined here and consider only loopless finite graphs. Let G be a graph. For each integer $i \geqslant 0$, define $V_i(G) = \{v \in V(G)\colon d_G(v) = i\}$ and $W(G) = V(G) \setminus V_2(G)$. As in [4], *a branch* in G is a nontrivial path whose end vertices are in $W(G)$ and whose internal vertices, if any, have degree 2 in G (and thus are not in $W(G)$). If a branch has length 1, then it has no internal vertices. We denote by $B(G)$ the set of branches of G and by $B_1(G)$ the subset of $B(G)$ in which every branch has an end vertex in $V_1(G)$. For any subgraph H of G, we denote by $B_H(G)$ the set of branches of G whose edges are all in H. For any two subgraphs H_1 and H_2 of G, define the *distance* $d_G(H_1, H_2)$ between H_1 and H_2 to be the minimum of the distances $d_G(v_1, v_2)$ over all pairs with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

The *line graph* of $G = (V(G), E(G))$ has $E(G)$ as its vertex set, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent (share an end vertex) in G. The *m-iterated line graph* $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^m(G) = L(L^{m-1}(G))$, where $L^1(G)$ denotes $L(G)$. The *hamiltonian index* of a graph G, denoted by $h(G)$, is the smallest integer m such that $L^m(G)$ is hamiltonian.

Chartrand [5] showed that if a connected graph G is not a path, then the hamiltonian index of G exists. In [6], a formula for the hamiltonian index of a tree other than a path was established.

There have already appeared many upper bounds on $h(G)$ in literature (see [4,6,8,11]). The following are the existing bounds that are rather easy to describe; the others involve more technical definitions and are omitted here.

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Theorem 1 (Lai [8]). *Let G be a connected simple graph that is not a path, and let l be the length of a longest branch of G which is not contained in a 3-cycle. Then $h(G) \leqslant l+1$.*

Theorem 2 (Saražin [11]). *Let G be a connected simple graph on n vertices other than a path. Then $h(G) \leqslant n - \Delta(G)$.*

Note that the graph in Theorem 2 must be simple, which is not mentioned in [11].

These known bounds are based on the following characterization of graphs with hamiltonian line graphs obtained in [7].

Theorem 3 (Harary and Nash-Williams [7]). *Let G be a graph with at least three edges. Then $h(G) \leqslant 1$ if and only if G has an eulerian subgraph H such that $d_G(e,H)=0$ for any edge $e \in E(G)$.*

Xiong and Liu [16] characterized the graphs for which the n-iterated line graph is hamiltonian, for any integer $n \geqslant 2$.

Theorem 4 (Xiong and Liu [16]). *Let G be a connected graph that is not a 2-cycle and let $n \geqslant 2$ be an integer. Then $h(G) \leqslant n$ if and only if $EU_n(G) \neq \emptyset$ where $EU_n(G)$ denotes the set of those subgraphs H of G which satisfy the following conditions*:

(i) *any vertex of H has even degree in H*;
(ii) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;
(iii) $d_G(H_1, H-H_1) \leqslant n-1$ *for any subgraph H_1 of H*;
(iv) $|E(b)| \leqslant n+1$ *for any branch b in $B(G) \setminus B_H(G)$*;
(v) $|E(b)| \leqslant n$ *for any branch in $B_1(G)$*.

Using Theorem 4, Xiong improved Theorem 2 as follows.

Theorem 5 (Xiong [15]). *Let G be a connected graph other than a path. Then $h(G) \leqslant \mathrm{dia}(G)-1$, where $\mathrm{dia}(G)$ denotes the diameter of G.*

From the vast and still growing pile of existing journal papers on this topic we deduce the importance of investigating whether the line graph of a graph is hamiltonian, i.e., whether $h(G) \leqslant 1$. Since from the above results it is clear that the line graph of a hamiltonian graph is again hamiltonian, the study on graphs with $h(G) \geqslant 2$ is equivalent to that on graphs with $h(G) \leqslant 1$. Motivated by these observations, and in an attempt to improve existing results including Theorem 1, we will give a reduction method to determine the hamiltonian index of a graph with $h(G) \geqslant 2$ in Section 3. Using this method we will give a sharp lower bound and a sharp upper bound on $h(G)$ such that the difference between the two bounds is exactly 2, in Section 4. Our results generalize results known earlier in [4,8,9,11,12].

In the Section 2, we will introduce the useful concept of a branch-bond, which will be applied throughout the paper, and is the basic idea behind the bounds.

2. Branch-bonds

For any subset S of $B(G)$, we denote by $G-S$ the subgraph obtained from $G[E(G) \setminus E(S)]$ by deleting all internal vertices of degree 2 in any branch of S. A subset S of $B(G)$ is called a *branch cut* if $G-S$ has more components than G. A minimal branch cut is called a *branch-bond*. If G is connected, then a branch cut S of G is a minimal subset of $B(G)$ such that $G-S$ is disconnected. It is easily shown that, for a connected graph G, a subset S of $B(G)$ is a branch-bond if and only if $G-S$ has exactly two components. We denote by $BB(G)$ the set of branch-bonds of G. Given $S,T \subseteq V(G)$, we write $[S,T]$ for the set of edges having one end vertex in S and the other in T. An *edge cut* is an edge set of the form $[S,\bar{S}]$, where S is a nonempty proper subset of $V(G)$ and $\bar{S}=V(G)\setminus S$. A minimal edge cut of G is called a *bond*. Obviously, if every branch in a branch-bond of G is an edge then the branch-bond is also a bond of G. The following characterization of eulerian graphs is well known [10].

Theorem 6 (McKee [10]). *A connected graph is eulerian if and only if each bond contains an even number of edges.*

The following characterization of eulerian graphs involving branch-bonds follows easily from Theorem 6.

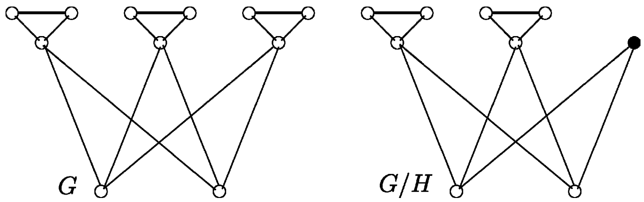


Fig. 1. A graph G with $h(G)=2$ and $h(G/H)=1$.

Theorem 7. *A connected graph is eulerian if and only if each branch-bond contains an even number of branches.*

3. A reduction method for determining the hamiltonian index of a graph

Before presenting our main results, we first introduce some additional notation.

Catlin [3] developed a reduction method for determining whether a graph G has a spanning closed trail. This method needs a tool, the so-called graph contraction. Let G be a graph and let H be a subgraph of G. We will give and use a refinement of Catlin's reduction method. The *contraction* of H in G, denoted by G/H, is the graph obtained from G by contracting all edges of H, i.e., replacing H by a new vertex v_H, which is called the *contracted vertex* in G/H, such that the number of edges in G/H joining any $v \in V(G) \setminus V(H)$ to v_H in G/H equals the number of edges joining v to H in G. Note that contractions may also result in loops and multiple edges, but that loops do not take place if the subgraph H is an induced subgraph of G (induced by a vertex subset). The following lemma follows from Theorem 7 and is needed for our proofs of the main results.

Lemma 8. *If G is an eulerian graph and H is a subgraph of G, then G/H is also an eulerian graph.*

Catlin's reduction method and Theorem 3 are useful in the study of the hamiltonian index, as seen in [4,8,11,12]. However, if we want to study the hamiltonicity of $L^m(G)$ for some $m \geqslant 1$ we must consider the lower iterated line graph $L^{m-1}(G)$ when we want to apply these results. If we use Theorem 4, then we might be able to avoid this (see [14,15,17]).

The graph G/H may have a smaller hamiltonian index than G. For example consider the graph G obtained from a $K_{2,3}$ by replacing each vertex of degree 2 in the $K_{2,3}$ by a triangle K_3, the graph G is depicted in Fig. 1. Let H be one of these newly added K_3's. Then G/H has an eulerian subgraph containing all vertices except the vertex representing H. Therefore, G/H has hamiltonian index 1 but clearly G has hamiltonian index 2.

In order to use Theorem 4 we have to assure that the property $h(G) \geqslant 2$ is preserved after the graph has been contracted. We do this by means of the attachment of new branches at the contracted vertex. Precisely, the *attachment–contraction* $G//H$ is the graph obtained from G/H by attaching two new edge-disjoint branches b'_H, b''_H of length two at the contracted vertex v_H (i.e., v_H is one end vertex of both b'_H and b''_H) such that b'_H, b''_H belong to $B_1(G//H)$. If $H_1, H_2, \ldots, H_k$ are vertex-disjoint nontrivial connected subgraphs of G, then $G//\{H_1, H_2, \ldots, H_k\}$ is obtained from G by *attachment–contracting* every H_i $(i = 1, 2, \ldots, k)$. For b_1 in $B(G)$ and b_2 in $B(G//\{H_1, H_2, \ldots, H_k\})$, we say $b_1 = b_2$ if b_1 contains all the internal vertices of b_2, and if the end vertices of b_1 are either the same as those of b_2 or they belong to $H_1 \cup H_2 \cup \cdots \cup H_k$.

Now we can state the main result of this section.

Theorem 9. *Let G be a connected graph other than a path, and let $G_1, \ldots, G_k$ be all nontrivial components of* $G[\{v : d_G(v) \geqslant 3\}] - \{e : e$ is a nontrivial cut edge of $G\}$. *If $h(G) \geqslant 2$, then*

$$h(G) = h(G//\{G_1, G_2, \ldots, G_k\}).$$

Proof. Let $G' = G//\{G_1, G_2, \ldots, G_k\}$. The following claim is straightforward.

Claim 1. *G and G' have the same branch set of length at least 2 and the same nontrivial cut edges set, except that* $\{b'_{G_1}, b''_{G_1}, b'_{G_2}, b''_{G_2}, \ldots, b'_{G_k}, b''_{G_k}\} \subseteq B_1(G') \setminus B(G)$.

First, we will prove that $h(G') \leqslant h(G)$. Take $H \in EU_{h(G)}(G)$. By (ii) in the definition of $EU_{h(G)}(G)$, H contains all vertices of $\bigcup_{i=1}^{k} V(G_i)$. We set $H_i = H[V(G_i)]$ for $i \in \{1, 2, \ldots, k\}$ and let $H' = H/\{H_1, H_2, \ldots, H_k\}$. Obviously H' is a subgraph of G' and H' contains all vertices of $\{v_{G_1}, v_{G_2}, \ldots, v_{G_k}\}$. We will prove that $H' \in EU_{h(G)}(G')$, which implies that $h(G') \leqslant h(G)$. Since H satisfies (i), H is a union of eulerian subgraphs in G. Hence it follows from Lemma 8 that H' is also a union of eulerian subgraphs of G', which implies that H' satisfies (i). That (ii) holds for H implies that (ii) holds for H' as well. In order to prove that H' satisfies (iii), it suffices to consider a subgraph $K' \subseteq H'$ with $d_{G'}(K', H' - K') \geqslant 2$. Let $K = H[V'_K \cup V''_K]$, where $V'_K = V(K') \cap V(G)$ and $V''_K = V(K') \cap \{v_{G_1}, v_{G_2}, \ldots, v_{G_k}\}$ is a set of contracted vertices. One can easily see that K is a subgraph of H and any shortest path P in G from K to $H - K$ has end vertices of degree at least 3 in G. So $P' = G'[E(P) \cap E(G')]$ is a path from K' to $H' - K'$ in G'. Hence, since H satisfies (iii), $d_{G'}(K', H' - K') \leqslant |E(P')| \leqslant |E(P)| = d_G(K, H - K) \leqslant h(G) - 1$. So H' satisfies (iii). By Claim 1, H' satisfies both (iv) and (v). Hence $H' \in EU_{h(G)}(G')$ which implies that $h(G') \leqslant h(G)$.

It remains to prove that $h(G) \leqslant h(G')$. Obviously $h(G') \geqslant 2$. By Theorem 4, there is a subgraph $H' \in EU_{h(G')}(G')$. We will construct a subgraph in $EU_{h(G')}(G)$ from H'. Since H' satisfies (ii), and by the definition of $G_1, G_2, \ldots, G_k$, H' contains all vertices of $\{v_{G_1}, v_{G_2}, \ldots, v_{G_k}\}$.

Set

$$V_{bi}(H') = \{x \in V(G_i) : x \text{ is an end vertex of a branch of } B_{H'}(G')\}$$

for $i \in \{1, 2, \ldots, k\}$ and

$$V_b = \bigcup_{i=1}^{k} V_{bi}(H).$$

We denote by $R(x)$ the number of branches of $B_{H'}(G')$ that have x as an end vertex. Set

$$V_{bi}^j = \{x \in V_{bi}(H') : R(x) \equiv j (\mathrm{mod}\, 2)\}$$

and

$$V_b^j = \bigcup_{i=1}^{k} V_{bi}^j \quad \text{for } j \in \{0, 1\}.$$

Since H' satisfies (i),

$$\sum_{x \in V_{bi}^0} R(x) + \sum_{x \in V_{bi}^1} R(x) = \sum_{x \in V_{bi}} R(x) = d_{H'}(v_{G_i})$$

is even. Since $\sum_{x \in V_{bi}^0} R(x)$ is even, it follows that $\sum_{x \in V_{bi}^1} R(x)$ is also even. Thus $|V_{bi}^1|$ is even.

Without loss of generality, assume

$$V_{bi}^1 = \{u_1^i, v_1^i, u_2^i, v_2^i, \ldots, u_{s_i}^i, v_{s_i}^i\}.$$

Since G_i is connected, we can select a shortest path, denoted by $P(u_j^i, v_j^i)$, between u_j^i and v_j^i in G_i for $j \in \{1, 2, \ldots, s_i\}$. Set

$$P(V_{\mathrm{b}}^1) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{s_i} \{P(u_j^i, v_j^i)\}.$$

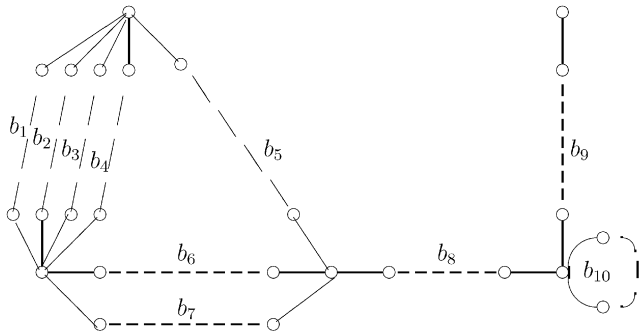
Let H be the subgraph of G with the following vertex set:

$$V(H) = \left(\bigcup_{i=1}^{k} V(G_i) \right) \bigcup \, (V(H') \setminus \{v_{G_1}, v_{G_2}, \ldots, v_{G_k}\})$$

and edge set

$$E(H) = E(H') \bigcup \{e \in \bigcup_{i=1}^{k} E(G_i) : |\{P \in P(V_b^1) : e \in E(P)\}| \equiv 1 (\mathrm{mod}\, 2)\}.$$

We will prove that $H \in EU_{h(G')}(G)$. First we prove that H satisfies (i).



$$BB_1(G) = \{\{b_9\}\}, BB_2(G) = \{\{b_8\}\},$$
$$BB_3(G) = \{\{b_1, b_2, b_3, b_4, b_5\}, \{b_5, b_6, b_7\}\}$$

Fig. 2. The definition of $\mathrm{BB}_i(G)$ illustrated.

Defining $E_x(G) = \{e \in E(G) \colon e$ is an edge that is incident with $x\}$, we have

$$d_{(\bigcup_{i=1}^{k} G_i) \bigcap H}(x) = \begin{cases} 2|\{P \in P(V_b^1) : x \in V(P)\}| - 1 - \sum\limits_{e \in E_x(G)} 2\lfloor \frac{1}{2} |\{P \in P(V_b^1) : e \in E(P)\}| \rfloor & \text{if } x \in V_b^1 \\ 2|\{P \in P(V_b^1) : x \in V(P)\}| - \sum\limits_{e \in E_x(G)} 2\lfloor \frac{1}{2} |\{P \in P(V_b^1) : e \in E(P)\}| \rfloor & \text{if } x \in V(G) \setminus V(V_b^1). \end{cases}$$

Hence, for any vertex $x \in V(H) \bigcap (\bigcup_{i=1}^{k} V(G_i))$, we have that

$$d_H(x) = d_{(\bigcup_{i=1}^{k} G_i) \bigcap H}(x) + R(x)$$

is even. For any $x \in V(H) \setminus (\bigcup_{i=1}^{k} V(G_i))$, we have $d_H(x) = d_G(x) = 2$. So H satisfies (i). Since H' satisfies (ii), H satisfies (ii). By Claim 1, H satisfies both (iv) and (v).

In order to prove that H satisfies (iii), we only need to consider a subgraph K of H such that $d_G(K, H-K) \geqslant 2$, since $h(G) \geqslant 2$. Hence, since

$$V(G_i) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H) \text{ for } i \in \{1, 2, \ldots, k\},$$

$V(K) \cap V(G_i)$ is either empty or $V(G_i)$ for $i \in \{1, 2, \ldots, k\}$. Let $K_1, K_2, \ldots, K_c$ be all nontrivial components of

$$K[\{v : d_K(v) \geqslant 3\}] - \{e : e \text{ is a cut edge of } G\}.$$

We obtain that $K' = K/\{K_1, K_2, \ldots, K_c\}$ is a subgraph of H'. Let $P' = x'u_1u_2 \ldots u_t y'$ be a shortest path from K' to $H' - K'$ in G'. Since $\{v_{G_1}, v_{G_2}, \ldots, v_{G_k}\} \subseteq V(H')$,

$$\{u_1, u_2, \ldots, u_t\} \cap \{v_{G_1}, v_{G_2}, \ldots, v_{G_k}\} = \emptyset.$$

Hence $\{u_1, u_2, \ldots, u_t\} \subseteq V(G)$. By the selection of K' and H, there exist two vertices $x \in V(K)$ and $y \in V(H-K)$ such that $xu_1, u_t y \in E(G)$. Hence $P = xu_1u_2 \ldots u_t y$ is a path from K to $H - K$, which implies that

$$d_G(K, H-K) \leqslant |E(P)| = |E(P')| = d_{G'}(K', H'-K') \leqslant h(G') - 1$$

and (iii) holds. By Theorem 4, $h(G) \leqslant h(G')$ which completes the proof of Theorem 9.

4. Sharp upper and lower bounds for $h(G)$

A branch-bond is called *odd* if it consists of an odd number of branches. The *length* of a branch-bond $S \in \mathrm{BB}(G)$, denoted by $l(S)$, is the length of a shortest branch in it. Define $\mathrm{BB}_2(G) = \{S \in \mathrm{BB}(G) : |S| = 1$ and both endvertices of $b \in S$ have degree $\geqslant 3$ in $G\}$ and $\mathrm{BB}_3(G) = \{S \in \mathrm{BB}(G) : |S| \geqslant 3$ and $|S|$ is odd$\}$. For convenience, we denote $\mathrm{BB}_1(G) = B_1(G)$. See Fig. 2 for an example.

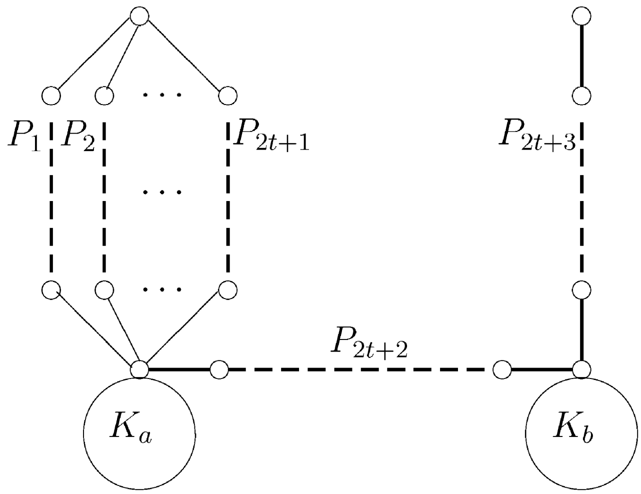


Fig. 3. An extremal graph with equality in (1).

Define

$$h_i(G) = \begin{cases} \max\{l(S) : S \in \mathrm{BB}_i(G)\} & \text{for } i \in \{1,2,3\} \quad \text{if } \mathrm{BB}_i(G) \text{ is not empty,} \\ 0 & \text{otherwise.} \end{cases}$$

If F_1 and F_2 are two subsets of $E(G)$, then $H + F_1 - F_2$ denotes the subgraph of G obtained from $G[(E(H) \bigcup F_1) \setminus F_2]$ by adding the remaining vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ as isolated vertices in $H + F_1 - F_2$.

The following lower bound for $h(G)$ involving odd branch-bonds can now be given.

Theorem 10. *Let G be a connected graph with $h(G) \geqslant 1$. Then*

$$h(G) \geqslant \max\{h_1(G), h_2(G) + 1, h_3(G) - 1\}. \tag{1}$$

Proof. If $h(G) = 1$, then, by Theorem 3, $h_1(G) \leqslant 1, h_2(G) \leqslant 0$ and $h_3(G) \leqslant 2$, i.e., (1) is true. So we can assume that G is a connected graph with $h(G) \geqslant 2$. We can take an $S_i \in \mathrm{BB}_i(G)$ such that $h_i(G) = l(S_i)$ for every $i \in \{1,2,3\}$. For any subgraph $H \in EU_{h(G)}(G), E(b) \cap E(H) = \emptyset$ for any $b \in S_1 \cup S_2$ and there exists at least a branch $b \in S_3$ such that $E(b) \cap E(H) = \emptyset$. Hence by Theorem 4, we obtain $h(G) \geqslant h_1(G)$ by (v), $h(G) \geqslant h_2(G) + 1$ by (iii) and $h(G) \geqslant h_3(G) - 1$ by (iv). So $h(G) \geqslant \max\{h_1(G), h_2(G) + 1, h_3(G) - 1\}$, i.e., (1) holds. □

We can construct an extremal graph for the equality (1). For an integer $t \geqslant 1$, we let $P_1, P_2, \ldots, P_{2t+3}$ be $2t + 3$ vertex-disjoint paths and let K_a, K_b be two vertex-disjoint complete graphs of order at least 3. Taking two fixed vertices in $V(K_a)$ and $V(K_b)$, respectively, we construct a graph G_0 by identifying exactly one end vertex of each of $P_1, P_2, \ldots, P_{2t+1}$ respectively, and identifying the other end vertex of $P_1, P_2, \ldots, P_{2t+1}$ with the fixed vertex from $V(K_a)$ and exactly one end vertex of P_{2t+2}; moreover we identify the fixed vertex of $V(K_b)$ with the other end vertex of P_{2t+2}, and exactly one end vertex of P_{2t+3}, such that $P_1, P_2, \ldots, P_{2t+3}, K_a, K_b$ are edge-disjoint subgraphs of G_0, as shown in Fig. 3.

Set

$$k_1(G_0) = \max\{h_1(G_0), h_2(G_0) + 1, h_3(G_0) - 1\}.$$

Obviously $h_1(G_0) = |E(P_{2t+3})|, h_2(G_0) = |E(P_{2t+2})|$ and

$$h_3(G_0) = \min\{|E(P_1)|, |E(P_2)| \ldots |E(P_{2t+1})|\}.$$

Without loss of generality, we assume that $|E(P_1)| = h_3(G)$. One can easily see that $K_a \cup K_b \cup (\bigcup_{i=2}^{2n+1} P_i) \in EU_{k_1(G_0)}(G_0)$. By Theorem 4, $h(G_0) \leqslant k_1(G_0)$. By (1), $h(G_0) \geqslant k_1(G_0)$. So $h(G_0) = \max\{h_1(G_0), h_2(G_0) + 1, h_3(G_0) - 1\}$.

Now we state our upper bound for $h(G)$.

Theorem 11. *Let G be a connected graph that is not a path. Then*

$$h(G) \leqslant \max\{h_1(G), h_2(G)+1, h_3(G)+1\}. \tag{2}$$

Proof. Let $k(G) = \max\{h_1(G), h_2(G)+1, h_3(G)+1\}$. Obviously $k(G) \geqslant 1$.

If $k(G) = 1$, i.e., $h_1(G) \leqslant 1$ and $h_2(G) = h_3(G) = 0$, then, by Theorem 7, $G[V(G) \setminus V_1(G)]$ is eulerian. Hence, using Theorem 3, we obtain $h(G) \leqslant 1$, i.e., (2) is true.

So we assume that $h(G) \geqslant 2$ and $k(G) \geqslant 2$. By Theorem 9, it suffices to consider the graph G such that $G[\{v : d_G(v) \geqslant 3\}] - \{e : e$ is a nontrivial cut edge of $G\}$ has no nontrivial component. Let $H \in EU_{h(G)}(G)$ be the subgraph with the maximal number of branches $b \in B_H(G)$ not contained in $\mathrm{BB}_1(G) \cup \mathrm{BB}_2(G)$ that have the property $|E(b)| \geqslant k(G)$. Then we can prove the following: □

Claim 1. *If S is a branch-bond in* $\mathrm{BB}(G)$ *such that it contains at least three branches, then there exists no branch $b \in S \setminus B_H(G)$ such that $|E(b)| \geqslant k(G)$.*

Proof of Claim 1. Otherwise there exists a branch $b_0 \in B(G) \setminus B_H(G)$ and a branch-bond S with $|S| \geqslant 3$ such that $|E(b_0)| \geqslant k(G)$ and $b_0 \in S \setminus B_H(G)$. Obviously b_0 has two end vertices u and v (say). Now we can select a branch-bond, denoted by $S(u, b_0)$, such that it contains b_0 and any branch of $S(u, b_0)$ has the end u. Obviously $|S(u, b_0)| \geqslant 2$.

In order to obtain a contradiction, we will first find a cycle of G that contains b_0 by the following algorithm.

Algorithm b_0.

1. If $|S(u, b_0)| \equiv 0 (\mathrm{mod}\, 2)$, then (by Theorem 7, we can) select a branch $b_1 \in S(u, b_0) \setminus (B_H(G) \cup \{b_0\})$. Otherwise (since $|E(b_0)| \geqslant k(G)$, we can) select a branch $b_1 (\neq b_0) \in S(u, b_0)$ with $|E(b_1)| = l(S(u, b_0)) \leqslant h_3(G) \leqslant k(G) - 1$ and let $u_1 (\neq u)$ be the other end vertex of b_1. If $u_1 = v$, then set $t := 1$ and stop. Otherwise $i := 1$.
2. Select a branch-bond $S(u, u_i, b_0)$ in G which contains b_0 but not $b_1, b_2, \ldots, b_i$ such that any branch in $S(u, u_i, b_0)$ has exactly one end vertex in $\{u, u_1, u_2, \ldots, u_i\}$. If $|S(u, u_i, b_0)| \equiv 0 (\mathrm{mod}\, 2)$, then (by Theorem 7, we can) select a branch
$$b_{i+1} \in S(u, u_i, b_0) \setminus (B_H(G) \cup \{b_0\}).$$
Otherwise (since $|E(b_0)| \geqslant k(G)$, we can) select a branch $b_{i+1} \in S(u, u_i, b_0)$ such that $b_{i+1} \neq b_0$ and $|E(b_{i+1})| = l(S(u, u_i, b_0)) \leqslant h_3(G) \leqslant k(G) - 1$, and let u_{i+1} be the end vertex of b_{i+1} that is not in $\{u, u_1, u_2, \ldots, u_i\}$.
3. If $u_{i+1} = v$, then set $t := i + 1$ and stop. Otherwise replace i by $i + 1$ and return to step 2.

Since $|B(G)|$ is finite and $d_G(v) \geqslant 2$, Algorithm b_0 will stop after a finite number of steps. Obviously, $G[\bigcup_{i=0}^{t} E(b_i)]$ is connected. Furthermore, since $u_t = v$ and $|S(u, u_i, b_0)| \geqslant 2, b_0$ is in a cycle of $G[\bigcup_{i=0}^{t} E(b_i)]$. Hence we obtain the following:

Claim 2. *$G[\bigcup_{i=0}^{t} E(b_i)]$ has a cycle of G, denoted by C_0, which contains b_0.*

Now we construct a subgraph $H' \subseteq G$ as follows:

$$H' = H + E(C_0) \setminus E(H) - (E(H) \cap E(C_0)).$$

By the selection of $\{b_1, b_2, \ldots, b_t\}$,

$$|E(b)| \leqslant h_3(G) \leqslant k(G) - 1 \quad \text{for} \quad b \in B_H(G) \cap \{b_1, b_2, \ldots, b_t\}.$$

Hence, by Claim 2, H' satisfies (iii) and (iv). Obviously H' satisfies (i), (ii) and (v), and this implies that H' is in $EU_{h(G)}(G)$. But H' contains b_0 which contradicts the maximality of H, which completes the proof of Claim 1.

For any branch b of G, if $G[E(b)]$ is not a cycle of G, then there exists a branch-bond $S \in \mathrm{BB}(G)$ with $b \in S$. Hence, by Claim 1 and the selection of $k(G), H \in EU_{k(G)}(G)$ which implies that $h(G) \leqslant k(G)$. The proof of Theorem 11 is completed.

We can construct a family of extremal graphs for Theorem 11. In fact, the following construction shows that $h(G_0)$ can take all integer values from $h_3(G_0) - 1$ to $h_3(G_0) + 1$. Let $k \geqslant 1$ be an integer and let $H = K_{2,2k+1}$ be a complete bipartite graph with $V^1(H) = \{x, y\}$ and $V^2(H) = \{u_1, u_2, \ldots, u_{2k+1}\}$. Let $1 \leqslant l_1 \ll l_2 \leqslant l_3 \ll l_4$ be four integers. The graph G_0 is obtained from H by subdividing $xu_1, xu_2, \ldots, xu_{2k}$ into $2k$ branches of length l_4, $yu_1, yu_2, \ldots, yu_{2k}$ into $2k$ branches of length l_1, xu_{2k+1} into a branch b of length l_2, yu_{2k+1} into a branch b' of length l_3, respectively, and by replacing each vertex of $V^2(H)$ by a K_4. See Fig. 4.

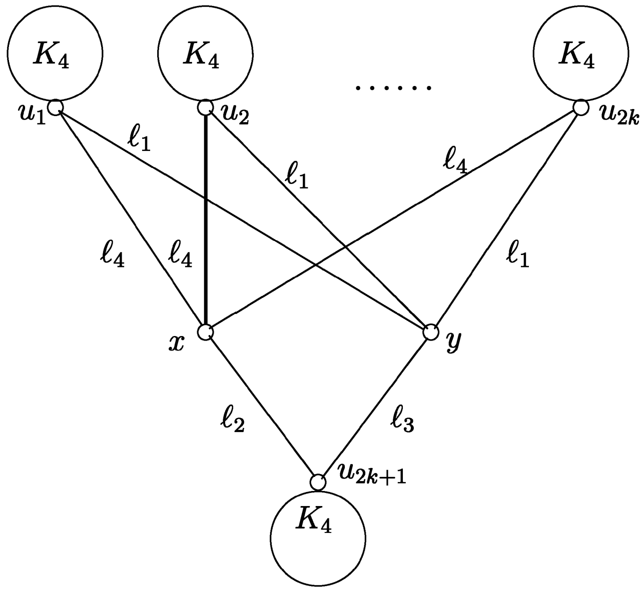


Fig. 4. An extremal graph with equality in (2).

One can easily see that $h_3(G_0) = l_3$ and that $G_0[E(G_0) \setminus (E(b) \cup E(b'))]$ has a subgraph in $EU_{\max\{l_3-1,l_2+1\}}(G_0)$. By Theorem 4, $h(G_0) \leqslant \max\{l_3-1, l_2+1\}$. By an argument similar to the one in the proof of (1), $h(G_0) \geqslant \max\{l_3-1, l_2+1\}$. Hence $h(G_0) = \max\{l_3 - 1, l_2 + 1\}$. Clearly

$$h(G_0) = \max\{l_3 - 1, l_2 + 1\} = \begin{cases} l_3 - 1 & \text{if } l_2 \leqslant l_3 - 2, \\ l_3 & \text{if } l_2 = l_3 - 1, \\ l_3 + 1 & \text{if } l_2 = l_3. \end{cases}$$

Hence $h(G_0)$ can take all integer values from $h_3(G_0) - 1$ to $h_3(G_0) + 1$ according to different integers l_2 and l_3.

Remark 12. It is easy to determine $h_1(G)$ and $h_2(G)$ of a graph G. Moreover, Woeginger [13] shows that there is a polynomial time algorithm to determine the parameter $h_3(G)$ of a graph G. Hence there are some polynomial time algorithms to determine these two bounds in above two theorems.

Although the graph G_0 above can reach all integer values from $h_3(G_0) - 1$ to $h_3(G_0) + 1$, it would be still interesting to give characterizations of graphs G whose hamiltonian indices are $h_3(G) - 1, h_3(G), h_3(G) + 1$, respectively. Hence we pose the following:

Question 13. *How to characterize those graphs G with $h(G) = h_3(G) - 1, h_3(G), h_3(G) + 1$, respectively?*

5. Analysis of known results

Theorems 5 and 11 show two upper bounds for the hamiltonian index of a graph. Clearly, $h_i(G) \leqslant \mathrm{dia}(G)$ for $i \in \{1, 2, 3\}$ and there exists a graph with large diameter and small $h_3(G)$, for example, the graph obtained by replacing each edge of a path by an odd branch-bond which contains at least three branches. Hence, the upper bound in Theorem 5 is not better than the one in Theorem 11. However, even the reverse does not necessarily hold true. Consider the graph F_t obtained from $K_{2,2t+1}$ by replacing each branch of $K_{2,2t+1}$ by a branch of lengths $\geqslant 2$. Clearly $h_3(F_t) = s = \mathrm{dia}(F_t)$ but $h(F_t) = s - 1 = h_3(G) - 1 = \mathrm{dia}(F_t) - 1$. The following relation between the two bounds in Theorems 5 and 11 is obtained.

Theorem 14. *Let G be a connected graph other than a path with $h(G) \geqslant 1$. If $h_3(G) = \mathrm{dia}(G)$, then*

$$h(G) = \mathrm{dia}(G) - 1.$$

Proof. This follows easily from Theorems 5 and 10. □

It would be interesting to give the characterization of graphs G with $h(G) = \mathrm{dia}(G) - 1$. Theorem 14 shows that it would relate to the parameter $h_3(G)$. Hence we pose the following:

Question 15. *How to use $h_3(G)$ to characterize those simple graphs G with $h(G) = \mathrm{dia}(G) - 1$?*

Obviously Theorem 1 is a consequence of Theorem 11. Although Theorem 2 is not a consequence of Theorem 11, one easily checks that $h_i(G) \leqslant n - \Delta(G)$ for any $i \in \{1,2,3\}$ and for a simple graph G, where $n = |V(G)|$. Hence, from Theorem 11, we have that $h(G) \leqslant n - \Delta(G) + 1$. Moreover, if $h_3(G) < n - \Delta(G)$, obviously, $\Delta(G) \leqslant n - (h_2(G) + 2)$, then Theorem 11 is better than Theorem 2. We pose the following:

Question 16. *How to use $h_1(G)$, $h_3(G)$ to characterize those simple graphs G with $h(G) = n - \Delta(G)$?*

Obviously $h_3(G) - 1 \leqslant h(G) \leqslant h_3(G) + 1$ for any 2-connected graph G. Hence we pose the following

Question 17. *How to use $h_3(G)$ to characterize those simple 2-connected graphs G with $h(G) = n - \Delta(G)$?*

The following result is an attempt to answer Question 17.

Theorem 18. *If G is a simple 2-connected graph with $h(G) = n - \Delta(G)$, then $h(G) \leqslant h_3(G) + 1 \leqslant 3$.*

Proof. If $h_3(G) \geqslant 3$, then $\Delta(G) \leqslant n - (3(h_3(G) - 2) + 2)$. Hence $h_3(G) \leqslant (n - \Delta(G) + 4)/3$ and $\Delta(G) \leqslant n - 5$. This implies that $h_3(G) \leqslant (n - \Delta(G) + 4)/3 < n - \Delta(G) - 1$. By Theorem 11, $h(G) \leqslant h_3(G) + 1 < n - \Delta(G)$, a contradiction. So $h_3(G) \leqslant 2$. Hence $h(G) \leqslant h_3(G) + 1 \leqslant 3$. □

The following consequences of Theorem 11 are easily obtained.

Corollary 19 (Catlin et al. [4]). *Let G be a connected graph that is neither a path nor a 2-cycle. Then*

$$h(G) \leqslant \max_{\{u,v\} \subseteq W(G)} \min_P X(P) + 1,$$

where $X(P)$ denotes the length $|E(b)|$ of the longest branch b in $B_P(G)$ and P is a subgraph induced by all branches in G whose end vertices are u and v.

Proof. Let S be a branch-bond in $BB(G)$ with $l(S) = \max\{h_1(G), h_2(G) + 1, h_3(G) + 1\}$. Then any path of G between two vertices u and v in two components of $G - S$, respectively, must have a branch in S. Hence

$$\max\{h_1(G), h_2(G) + 1, h_3(G) + 1\} \leqslant \max_{\{u,v\} \subseteq W(G)} \min_P X(P) + 1.$$

This relation and Theorem 11 give Corollary 19. □

Corollary 20 (Chartrand and Wall [6]). *If T is a tree which is not a path, then*

$$h(T) = \max\{h_1(T), h_2(T) + 1\}.$$

Proof. If T is a tree, then $h_3(T) = 0$. Hence by Theorems 10 and 11, we obtain Corollary 20. □

Corollary 21 (Balakrishnan and Paulraja [1]). *Let G be a connected graph with at least four edges. If the only 2-degree cut sets of G are the singleton subsets which are neighbors of end vertices of G, then $h(G) \leqslant 2$.*

Proof. One can easily check that $h_1(G) \leqslant 2$, $h_2(G) \leqslant 1$ and $h_3(G) \leqslant 1$. Hence this corollary follows from Theorem 11. □

Corollary 22 (Lesniak-Foster and Williamson [9]). *Let G be a connected graph with at least four edges. If every vertex of degree two is adjacent to an end vertex, then $h(G) \leqslant 2$.*

Proof. From the condition of this corollary, we know $h_1(G) \leqslant 2$, $h_2(G) \leqslant 1$ and $h_3(G) \leqslant 1$. Hence this corollary follows from Theorem 11. $\square$

Corollary 23 (Chartrand and Wall [6]). *Let G be a connected graph other than a path. If* $\delta(G) \geqslant 3$, *then* $h(G) \leqslant 2$.

Proof. This is obvious. $\square$

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